Analytical Sensitivity for the Zero-D Earth Temperature Model

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I have expanded the look at sensitivity to include application of an analytical sensitivity approach to the simple Zero-D Earth Temperature model. A few numerical values are also given.

An equation for the radiative-equilibrium energy exchange for the Earth is almost always stated to be

$$\frac{1}{4}(1-\rho) f_{es} \varepsilon_{Sun} \sigma T_{sun}^4 = (1-\gamma) \varepsilon_{Erth} \sigma T_e^4 \tag{1.1}$$

where

ρ reflectivity of the ultra violet by the Earth atmosphere

γ reflectance of the infra red by the Earth surface

f_{es} Earth-Sun view factor

T_{sun} effective radiative temperature of the Sun surface

σ Stefan-Boltzmann constant

T_e effective temperature of the Earth surface

ε_{Sun} emissivity for the Sun surface

ε_{Erth} emissivity for the Earth surface

Some nominal values for the parameters are as follows

$\rho = 0.30$

$\gamma = 0.40$

$\varepsilon_{Erth} = 0.87$

$\varepsilon_{Sun} = 1.00$

$T_{sun} = 5780.0$

$R_{Sun} = 6.96\text{x}10^{08}$

$R_{ES} = 1.496\text{x}10^{11}$

$f_{es} = 2.1646\text{E-}05$

$L = 3.84194\text{E+}26$

Where L is the luminosity at the Sun's surface. The Earth-Sun view factor is

$$f_{es} = \left(\frac{R_{Sun}}{R_{ES}} \right)^2 \tag{1.2}$$

where R_{Sun} is the radius of the Sun, and R_{ES} is the Earth-Sun distance. Nominal values for these are

$$R_{Sun} = 6.960\text{x}10^8\,\text{m}$$
$$R_{ES} = 1\text{AU} = 1.496\text{x}10^{11}\,\text{m}$$

The luminosity for the Sun surface taken to be a perfect radiator is

$$L_{Sun} = 4\pi R_{Sun}^2 I_{Sun} = 4\pi R_{Sun}^2 \varepsilon_{Sun} \sigma T_{Sun}^4 \tag{1.3}$$

and the power hitting a unit perpendicular area the Earth surface is

$$I_{TOA} = \varepsilon_{Sun} \sigma T_{Sun}^4 \left(\frac{R_{Sun}}{R_{ES}} \right)^2 \tag{1.4}$$

Putting it all together, the temperature of the surface that represents the Earth is

$$T_e = \left[\frac{1}{4} \frac{\varepsilon_{Sun}(1-\rho)}{\varepsilon_{Erth}(1-\gamma)} \right]^{1/4} \left(\frac{R_{Sun}}{R_{ES}} \right)^{½} T_{Sun} \tag{1.5}$$

The Sensitivity Coefficients

We want to know how the Earth temperature, T_e varies as the parameters are varied. The approach has been outlined in a previous posts; http://models-methods-software.com/2009/03/21/analytical-sensitivity-analysis/ and http://models-methods-software.com/2010/01/02/interval-arithmetic/ . In the latter post interval arithmetic was used to get the range that T_e takes due to the uncertainties in the parameters. As in the former post, the sensitivity coefficient of the response function, R, to a parameter is

$$S_{\alpha i} = \left(\frac{\partial R}{\partial \alpha_i} \right)_{\alpha j} \tag{1.6}$$

where αi represents a parameter, and the subscript αj indicates that all other parameters are held constant. All the coefficients have different units, so the usual practice is to normalize the coefficients to a reference state,

$$\hat{S}_{\alpha i} = \frac{\alpha_i^0}{R^0}\left(\frac{\partial R}{\partial \alpha_i}\right)_{\alpha j} \tag{1.7}$$

The results of applying the method to Eq. (1.5) are summarized in the nearby Table

Parameter	**Normalized Sensitivity**	**Numerical Sensitivity**	T_{Erth} **Estimate**	T_{Erth} **Re-calc**	**Error** (%)
Reflectivity, ρ	$\frac{\rho}{4(\rho-1)}$	-103.47	294.9	291.3	-1.25
Reflectance, γ	$-\frac{\gamma}{4(\gamma-1)}$	120.72	283.7	287.4	1.28
Sun emissivity, ε_{Sun}	$\frac{1}{4}$	72.43	286.1	286.0	-0.024
Earth emissivity, ε_{Erth}	$-\frac{1}{4}$	-83.25	293.9	303.9	3.28
Sun radius, R_{Sun}	$\frac{1}{2}$	2.08 x 10^{-07}	289.7	282.4	-2.60
Earth-Sun distance, R_{ES}	$-\frac{1}{2}$	-9.68 x 10^{-10}	289.7	297.3	2.53
Sun temperature, T_{Sun}	1.0	0.050	289.7	275.2	-5.26

The numerical sensitivity values are based on the nominal values listed previously above. If nominal values for the emissivity of the Sun and Earth are taken to be near unity, application of increases might lead to values greater than unity, and that's not possible.

These are first-order linearization for the sensitivity of the response, T_e, with respect to the parameters. The accuracy when extrapolating perturbations in the parameters based on the first-order estimates might not always be very good.

The numerical sensitivity values are used as follows, taking the Earth reflectivity as an example. The sensitivity is

$$S_\rho = \left(\frac{T_{Erth}^0}{\rho^0}\right)\hat{S}_\rho \tag{1.8}$$

which is given in the Table 1 above. The estimated temperature of the Earth is given by

$$\left(\Delta T_{Erth}\right)=\left(\frac{T_{Erth}^{0}}{\rho^{0}}\right)\hat{S}_{\rho}\left(\Delta\rho\right) \tag{1.9}$$

along with

$$T_{Erth}\cong T_{Erth}^{0}+\left(\Delta T_{Erth}\right) \tag{1.10}$$

If a delta of -5.0% is applied one-at-a-time to the parameters, and the equation for the response, Eq. (1.5), is used to re-calculate the actual value, the error in the extrapolation can be determined. Those results are given in the last column in Table 1. The error is defined by

Error=(re-calculated value - sensitivity value)100 / re-calculated value

The re-calculated values for T_e range from 275.24 K to 303.86 K and the nominal value is 289.73 K. The former value is from extrapolation of the Sun temperature and the latter from the emissivity for the Earth.